VARIATIONAL SOLUTION OF THE AXISYMMETRIC PROBLEM OF THERMOELASTICITY

(VARIATSIONNOE RESHENIE OSESIMMETRICHNOI ZADACHI TERMOUPRUGOSTI)

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The solution of an equation of the type $A\varphi = f$, where the operator A is symmetric and positive, can be reduced to the problem on the minimal functional, whereby the solution of the latter always exists if the operator A is positive definite [1,2]. The theorem on the minimal functional [3] has been established under more general properties of the operator A. In case the solution of the equation is not unique, the operator A will not possess the mentioned properties. In order to exclude the nonuniqueness of the solution, one imposes additional conditions on the region of definition of the operator. These conditions are not always obvious or convenient for the given problem. In a number of cases, in particular in the axisymmetric problem of the theory of elasticity, the nonuniqueness of the solution is unessential for the problem.

In the present work, the theorem on the minimal functional, in the general form in which it was considered by Martyniuk [3], has been extended to the case when the solution is not unique. This has been done by generalizing in a certain sense the properties of the operator. Hence the predetermination of the domain of definition of the operator A has been avoided. For the sake of shortening the presentation, use has been made of [1,2], and the proofs of theorems similar to those given there have been omitted.

A theorem is used as the basis of the variational method which is used in the solution of the axisymmetric problem of thermoelasticity. This problem is reduced to the variational problem with the aid of the proof of the appropriate inequality. The mean error is determined for the approximate solutions. The problem on the axisymmetric deformation of a hollow cylinder of finite length under constant temperature is solved. A numerical example is considered.

1. Let H be a complete Hilbert space. On the linear set M, dense in H, there is given the equation

$$A\varphi = f, \qquad (\varphi \in M,\ f \in H) \tag{1.1}$$

where A is a linear (additive and homogeneous) operator defined on M.

Suppose there exists some linear operator B, such that the linear set of elements $g = B\varphi \in H$ if $\varphi \in M$. We shall assume that this set is dense in H. We construct the scalar product $(A\varphi, B\psi)$ (where φ, $\psi \in M$), and introduce the following definitions.

The operator A is said to be symmetric with the operator B (or simply symmetric) if

$$(A\varphi, B\psi) = (A\psi, B\varphi), \qquad (\varphi, \psi \in M) \tag{1.2}$$

The operator A is said to be positive with the operator B (or simply positive) if, in addition to (1.2)

$$(A\varphi, B\varphi) \geqslant 0 \qquad (\varphi \in M) \tag{1.3}$$

where the equality to zero is attained if, and only if, $B\varphi = 0$. Then the set of elements for which $A\varphi = 0$, and which determine nonunique solutions of equation (1.1) are included in the set of elements for which $B\varphi = 0$. It is assumed that the latter is not dense in H. Thus with the aid of the operator B one cannot distinguish the elements φ for which $B\varphi$ is the same; a zero element will be an element for which $B\varphi = 0$.

The following theorem corresponds to the newly introduced definitions. If the operator A is positive, then the solution of the equation (1.1) is "unique".

Assuming that φ_1 and φ_2 are two distinct solutions, $A\varphi_1 = f$, $A\varphi_2 = f$, and forming the scalar product of the form $(A\varphi, B\varphi)$ for the difference $\varphi = \varphi_1 - \varphi_2$ of the solutions, we find (due to the positiveness of the operator A) that it is necessary that $B\varphi_1 = B\varphi_2$, i.e. the solutions φ_1 and φ_2 are not distinct in the above sense. From this theorem it follows that the scalar products $(A\varphi_1, B\varphi_1)$ and $(A\varphi_2, B\varphi_2)$ are equal if $B\varphi_1 = B\varphi_2$.

We shall now formulate the theorem on the minimal functional.

Theorem. 1) If the operator A is positive, then the solution of the equation (1.1) yields the smallest value for the functional

$$J(\varphi) = (A\varphi, B\varphi) - 2(f, B\varphi) \tag{1.4}$$

2) Conversely, if φ yields a minimum of the functional (1.4) then φ is

a solution of the equation (1.1).

The existence of a solution of the variational problem, the problem of finding the minimum of the functional (1.4), is established in some special space under the assumption that the operator A is positive definite with the operator B.

The operator A is said to be positive definite with the operator B (or simply positive definite) if, in addition to (1.2)

$$(A\varphi, B\varphi) \geqslant \gamma^2 \|B\varphi\|^2, \quad \gamma > 0 \qquad (\varphi \in M) \tag{1.5}$$

For the elements of the set M, which are distinct in the specified sense, we define the scalar product by means of the formulas

$$(\varphi, \psi)_B = (B\varphi, B\psi), \qquad (\varphi, \psi)_A = (A\varphi, B\psi) \tag{1.6}$$

One can easily verify that these definitions satisfy the axioms of a scalar product [1].

Adding to the set M the limit elements according to the metric (1.6), we obtain complete Hilbert spaces, which we shall denote by H_B and H_A, respectively. The norms of the elements of these spaces are defined by the formulas

$$\|\varphi\|_B = \sqrt{(\varphi, \varphi)_B}, \qquad \|\varphi\|_A = \sqrt{(\varphi, \varphi)_A}$$

It is obvious that M is dense in H_B and in H_A. The connection between the spaces H_B and H_A, for elements belonging to M, is determined by (1.5),

$$\|\varphi\|_A \geqslant \gamma \|\varphi\|_B, \qquad \gamma > 0 \tag{1.7}$$

With the aid of (1.7) one can prove a theorem on the imbedding of the space H_A in H_B: with every element of H_A one can associate an element of H_B in such a manner that to distinct elements of H_A there correspond distinct elements of H_B. Hence the inequality (1.7) is extended over the entire space H_A.

The proof of the existence of a solution is based on the fact that for every element $f \in H$ we have, on the basis of the Cauchy-Buniakovskii formula, and by (1.7), the following inequality

$$|(f, B\varphi)| \leqslant \|f\| \|\varphi\|_B \leqslant \frac{\|f\|}{\gamma} \|\varphi\|_A, \qquad \varphi \in M$$

i.e. $(f, B\varphi)$ is a functional that is bounded in H_A. Hence by the theorem of Riesz [1] there exists a unique element φ_0 in H_A in terms of which one can express the functional in the form $(f, B\varphi) = (\varphi, \varphi_0)_A$. Assuming that $J(\varphi)$ is defined in the entire space H_A, and taking into account (1.6), we find that

$$J(\varphi) = (\varphi, \varphi)_A - 2(\varphi, \varphi_0)_A = \|\varphi - \varphi_0\|_A^2 - \|\varphi_0\|_A^2 \qquad (1.8)$$
$$\min J(\varphi) = -\|\varphi_0\|_A^2, \quad \text{for } \varphi = \varphi_0$$

From this it follows that the element which makes the functional (1.4) a minimum can not belong to M, the domain of definition of the operator A. In this case we have a unique solution in the extended domain of definition of the operator A, i.e. in the space H_A. To it there will correspond in the space H_B some limit element $g^* = B\varphi^* = \lim B\varphi_n$, $\varphi_n \in M$ (the limit is here understood in the sense $\|\varphi^* - \varphi_n\|_B \to 0$, $n \to \infty$). The solutions which do not belong to M will be called generalized solutions.

We now assume that the space H_A is separable, and we shall try to construct in it a solution of the variational problem. Let φ_n be a complete, orthonormalized sequence of elements in H_A, i.e.

$$(\varphi_n, \varphi_m)_A = 0, \quad \text{if} \quad n \neq m, \qquad \|\varphi_n\|_A = 1 \qquad (n = 1, 2 \ldots) \qquad (1.9)$$

An element φ_0 which might make the functional $J(\varphi)$ a minimum can now be expressed as an expansion in terms of the orthogonal functions

$$\varphi_0 = \sum_{n=1}^{\infty} (\varphi_0, \varphi_n)_A \varphi_n = \sum_{n=1}^{\infty} (f, B\varphi_n)\varphi_n \qquad (1.10)$$

If in the solution (1.10) one takes a finite number of terms of the series, then the sequence of elements

$$\varphi^m = (f, B\varphi_1)\varphi_1 + \ldots + (f, B\varphi_m)\varphi_m$$

will be a minimizing sequence since

$$J(\varphi^m) = \|\varphi^m - \varphi_0\|_A^2 - \|\varphi_0\|_A^2 \to -\|\varphi_0\|_A^2 = \min J(\varphi)$$

If we are given in M a system of elements ψ_n, which is linearly independent and complete in H_A, then substituting in (1.4) a linear combination of these elements

$$\psi^m = a_1\psi_1 + \ldots + a_m\psi_m \qquad (1.11)$$

we obtain $J(\psi^m)$ as a function of the parameters a_k. Formulating the conditions for a minimum of the function $J(a_k)$, we obtain a system for the determination of the coefficients

$$(A\psi_1, B\psi_n)\, a_1 + \ldots + (A\psi_m, B\psi_n)\, a_m = (f, B\psi_n) \qquad (n = 1,2,\ldots, m) \qquad (1.12)$$

One can show [1] that ψ^m is a minimizing sequence when $m \to \infty$ if the a_k are determined by the equations (1.12).

Simultaneously one establishes the convergence of the method of Ritz in the form (1.12) (or the method of squares Ritz [3]). Galerkin's method leads to the same type of system of equations for the determination of the a_k in (1.11)

$$(A\psi^m - f, B\psi_n) = 0 \qquad (n = 1,2, \ldots, m) \qquad (1.13)$$

Hence, this method will also converge if the operator A has the above specified properties. The equations (1.13) coincide with one of the generalized forms of Galerkin's method proved by several authors [4-6] under the assumption of the uniqueness of a solution.

In what follows we shall assume that the space H is the space of square-summable functions with some weight p (p is a positive function), i.e. H is an $L_2(p)$ space which has been shown [7] to be a complete, separable, Hilbert space.

2. The axisymmetric problem of thermoelasticity, for the case of a circular cylinder of finite length, can be reduced to the solution of an equation in the potential function $\varphi(\rho, \zeta)$

$$\frac{1}{\rho}\Delta_1^2\varphi = \frac{1}{\rho}\left(DD + 2\frac{\partial^2}{\partial\zeta^2}D + \frac{\partial^4}{\partial\zeta^4}\right)\varphi = \frac{1}{\rho}f(\rho,\zeta), \quad D = \rho\frac{\partial}{\partial\rho}\frac{1}{\rho}\frac{\partial}{\partial\rho} \qquad (2.1)$$

$$\left(\rho = \frac{r}{R}, \quad \zeta = \frac{z}{R}, \quad \frac{r_0}{R} = \rho_0 \leqslant \rho \leqslant 1, \quad 0 \leqslant \zeta \leqslant l = \frac{L}{R}\right)$$

Here, R is the outer radius of the cylinder; r_0 is the inner radius, L the length of the cylinder. The function $\varphi(\rho, \zeta)$ is determined by the formulas

$$\sigma_r = F_r + \frac{1}{\rho}\frac{\partial^3\varphi}{\partial\rho\partial\zeta^2} - \frac{1-\mu}{\rho^2}\frac{\partial^2\varphi}{\partial\zeta^2} + \frac{\mu}{\rho^2}D\varphi, \qquad \sigma_z = F_z + \frac{1}{\rho}\frac{\partial D\varphi}{\partial\rho}$$

$$\sigma_\theta = F_\theta + \frac{\mu}{\rho}\frac{\partial^3\varphi}{\partial\rho\partial\zeta^2} + \frac{1-\mu}{\rho^2}\frac{\partial^2\varphi}{\partial\zeta^2} + \mu\frac{\partial}{\partial\rho}\frac{D\varphi}{\rho}, \qquad \sigma_{rz} = F_{rz} - \frac{1}{\rho}\frac{\partial D\varphi}{\partial\zeta} \qquad (2.2)$$

and satisfies the boundary condition (the problem is solved for the stresses)

$$D\varphi = 0, \quad \frac{\partial^3\varphi}{\partial\rho\partial\zeta^2} = \frac{1-\mu}{\rho}\frac{\partial^2\varphi}{\partial\zeta^2} \quad \text{if} \quad \rho = \begin{cases}\rho_0\\ 1\end{cases}$$

$$D\varphi = 0, \quad \frac{\partial D\varphi}{\partial \zeta} = 0 \quad \text{if} \quad \zeta = \begin{cases} 0 \\ l \end{cases} \tag{2.3}$$

This can easily be verified by substituting (2.2) into the equations of equilibrium and into the condition of density, and by taking into account Hooke's law and temperature deformations [8], e.g. in the case when the cylinder is subjected only to the effect of temperature $t(\rho, \zeta)$, whose distribution in the cylinder is determined by the equation

$$\frac{\partial^2 t}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial t}{\partial \rho} + \frac{\partial^2 t}{\partial \zeta^2} = f_1(\rho, \zeta)$$

In this case

$$F_z = 0, \; F_{rz} = 0 \tag{2.4}$$

$$F_r = \frac{E\alpha}{1-\rho_0^2}\int_{\rho_0}^{1} t\rho d\rho \left(1 - \frac{\rho_0^2}{\rho^2}\right) - \frac{E\alpha}{\rho^2}\int_{\rho_0}^{\rho} t\rho d\rho$$

$$F_\theta = \frac{E\alpha}{1-\rho_0^2}\int_{\rho_0}^{1} t\rho d\rho \left(1 + \frac{\rho_0^2}{\rho^2}\right) + \frac{E\alpha}{\rho^2}\int_{\rho_0}^{\rho} t\rho d\rho - E\alpha t$$

$$f(\rho, \zeta) = \frac{E\alpha}{1-\rho_0^2}\left\{\frac{\rho^2}{1+\mu}\left[\frac{\partial t(1)}{\partial \rho} - \rho_0\frac{\partial t(\rho_0)}{\partial \rho}\right] + \right.$$

$$\left. + \frac{\rho_0^2}{1-\mu}\left[\frac{\partial t(1)}{\partial \rho} - \frac{1}{\rho_0}\frac{\partial t(\rho_0)}{\partial \rho}\right] - \left(\frac{\rho^2}{1+\mu} + \frac{\rho_0^2}{1-\mu}\right)\int_{\rho_0}^{1} f_1 \rho \, d\rho\right\} - \frac{E\alpha}{1-\mu}\int_{\rho_1}^{\rho} f_1 \, \rho d\rho$$

If on the right-hand side of the equation of heat conduction, and in (2.4), in place of $f_1(\rho, \zeta)$ one considers $\partial t/\partial \tau + f_2(\rho, \zeta)$, where τ is time, then one obtains the case of a quasistationary problem.

The domain of definition of the operator $A = \rho^{-1}\Delta_1^{\,2}$ will be the set M of functions four times differentiable with respect to ρ and ζ, and satisfying the boundary conditions (2.3). It has been proved [2] that such a set is dense in $L_2(p)$, where in the given case $p = \rho$.

We shall show that the operator $A = \rho^{-1}\Delta_1^{\,2}$ is positive definite with the operator $B = -\rho^{-1}D$.

The solution of the equation is not unique and can be determined to within $\varphi^0 = a_0 + a_2\rho^2 + (b_0 + b_2\rho^2)\zeta$. It is easily seen that φ^0 is the general expression of the zero element defined by $B\varphi^0 = 0$. The space H_B in which the solution is sought will be determined by the scalar product and the norm

$$(\varphi, \psi)_B = \int_0^l\int_{\rho_0}^1 \frac{1}{\rho} D\varphi D\psi \, d\rho d\zeta, \qquad \|\varphi\|_B^2 = \int_0^l\int_{\rho_0}^1 \frac{1}{\rho}(D\varphi)^2 \, d\rho d\zeta \tag{2.5}$$

Integrating by parts, and taking into account the condition (2.3), we obtain

$$(A\varphi, B\psi) = \int_0^l\int_{\rho_0}^1 \frac{1}{\rho}\left(\frac{\partial D\varphi}{\partial\rho}\frac{\partial D\psi}{\partial\rho} + 2\frac{\partial D\varphi}{\partial\zeta}\frac{\partial D\psi}{\partial\zeta} + \frac{\partial^3\varphi}{\partial\rho\partial\zeta^2}\frac{\partial^3\psi}{\partial\rho\partial\zeta^2}\right)d\rho d\zeta -$$
$$-(1-\mu)\int_0^l\left(\frac{\partial^2\varphi(1)}{\partial\zeta^2}\frac{\partial^2\psi(1)}{\partial\zeta^2} - \frac{1}{\rho_0^2}\frac{\partial^2\varphi(\rho_0)}{\partial\zeta^2}\frac{\partial^2\psi(\rho_0)}{\partial\zeta^2}\right)d\zeta \tag{2.6}$$

Interchanging φ and ψ in this equation, we find that $(A\varphi, B\psi) = (A\psi, B\varphi)$, i.e. the operator A is symmetric. From (2.6) we obtain

$$(A\varphi, B\varphi) = \int_0^l\int_{\rho_0}^1 \frac{1}{\rho}\left[\left(\frac{\partial D\varphi}{\partial\rho}\right)^2 + 2\left(\frac{\partial D\varphi}{\partial\zeta}\right)^2 + \left(\frac{\partial^3\varphi}{\partial\rho\partial\zeta^2}\right)^2\right]d\rho d\zeta -$$
$$-(1-\mu)\int_0^l\left[\left(\frac{\partial^2\varphi(1)}{\partial\zeta^2}\right)^2 - \frac{1}{\rho_0^2}\left(\frac{\partial^2\varphi(\rho_0)}{\partial\zeta^2}\right)^2\right]d\zeta \tag{2.7}$$

Let us establish the inequality

$$x_1^2 - \frac{1}{\rho_0^2}x_0^2 \leqslant \frac{1}{2}\int_{\rho_0}^1 \frac{1}{\rho}\left(\frac{\partial x}{\partial\rho}\right)^2 d\rho \tag{2.8}$$

Here $x = x(\rho)$, $x_1 = x(1)$, $x_0 = x(\rho_0)$, $0 \leqslant \rho_0 \leqslant 1$. Suppose that x_1 and x_0 have the same sign. Let us set $x_0 = kx_1$, $0 \leqslant k \leqslant 1$. Then, obviously

$$(1-k)x_1 = x_1 - x_0 = \int_{\rho_0}^1 \frac{\partial x}{\partial\rho}d\rho$$

Applying Hölder's inequality [9], we obtain

$$(1-k)^2 x_1^2 \leqslant \left(\int_{\rho_0}^1 \left(\frac{1}{\rho}\frac{\partial x}{\partial\rho}\right)^2 \rho d\rho\right)\left(\int_{\rho_0}^1 1^2 \rho d\rho\right) = \frac{1-\rho_0^2}{2}\int_{\rho_0}^1 \frac{1}{\rho}\left(\frac{\partial x}{\partial\rho}\right)^2 d\rho$$

Whence

$$x_1^2 - \frac{1}{\rho_0^2}x_0^2 \leqslant f(\rho_0, k)\int_{\rho_0}^1 \frac{1}{\rho}\left(\frac{\partial x}{\partial\rho}\right)^2 d\rho \qquad f(\rho_0, k) = \frac{(1-\rho_0^2)(\rho_0^2 - k^2)}{2\rho_0^2(1-k)^2}$$

It is easy to verify that $f(\rho_0, k)$ does not have an extremum in the region $0 < \rho_0 < 1$, $0 < k < 1$, whereas $f \equiv 1/2$ when $k = \rho_0^2$. Obviously, f will not have an extremum also on the single-values curve $k = \rho_0^n$, $n > 1$.

The function

$$f(\rho_0) = (1-\rho_0^2)(1-\rho_0^{2n-2})/2(1-\rho_0^n)^2$$

will then attain its largest and smallest values when $\rho_0 = 0$ and $\rho = 1$.

$$f(0) = 0, \qquad \lim_{\rho \to 1} f(\rho_0) = \frac{2(n-1)}{n^2} \leqslant \frac{1}{2}$$

If x_1 and x_0 have different signs, then by the assumed continuity of $x(\rho)$, there exists a value $\rho = \rho_a$ for which $x(\rho_a) = 0$. Then

$$x_1 = \int_{\rho_a}^{1} \frac{\partial x}{\partial \rho} d\rho, \qquad x_1^2 \leqslant \frac{1-\rho_a^2}{2} \int_{\rho_a}^{1} \frac{1}{\rho} \left(\frac{\partial x}{\partial \rho}\right)^2 d\rho \leqslant \frac{1}{2} \int_{\rho_0}^{1} \frac{1}{\rho} \left(\frac{\partial x}{\partial \rho}\right)^2 d\rho$$

This establishes the inequality (2.8).

From (2.8) it follows that

$$\int_0^l \left[\left(\frac{\partial^2 \varphi(1)}{\partial \zeta^2}\right)^2 - \frac{1}{\rho_0^2}\left(\frac{\partial^2 \varphi(\rho_0)}{\partial \zeta^2}\right)^2\right] d\zeta \leqslant \frac{1}{2} \int_0^l \int_{\rho_0}^{1} \frac{1}{\rho} \left(\frac{\partial^3 \varphi}{\partial \rho \partial \zeta^2}\right)^2 d\rho d\zeta \tag{2.9}$$

Making use of (2.9), one can now easily establish the positiveness of the operator A.

Let us omit from (2.7) the obviously positive terms, so that

$$(A\varphi, B\varphi) \geqslant \int_0^l \int_{\rho_0}^{1} \frac{1}{\rho} \left[\left(\frac{\partial D\varphi}{\partial \rho}\right)^2 + \left(\frac{\partial D\varphi}{\partial \zeta}\right)^2\right] d\rho d\zeta = \int_0^l \int_{\rho_0}^{1} \left[\left(\frac{\partial v}{\partial \rho}\right)^2 + \left(\frac{\partial v}{\partial \zeta}\right)^2\right] d\rho d\zeta, \quad v = \frac{D\varphi}{\sqrt{\rho}}$$

From (2.3) it now follows that: $\rho = \rho_0$, $\rho = 1$; $\zeta = 0$, $\zeta = l$; $v = 0$, i.e. on the perimeter of the rectangle $v = 0$.

Friedrichs' [10] inequality can now be applied and we obtain

$$\int_0^l \int_{\rho_0}^{1} \left[\left(\frac{\partial v}{\partial \rho}\right)^2 + \left(\frac{\partial v}{\partial \zeta}\right)^2\right] d\rho d\zeta \geqslant \gamma^2 \int_0^l \int_{\rho_0}^{1} v^2 d\rho d\zeta = \gamma^2 \int_0^l \int_{\rho_0}^{1} \frac{1}{\rho} (D\varphi)^2 d\rho d\zeta$$

This shows that the operator A is positive definite (1.5).

Thus, the solution of equation (2.10) can be determined as the solution found from the condition that the functional

$$J(\varphi) = -\int_0^l \int_{\rho_0}^{1} \frac{1}{\rho} \Delta_1^2 \varphi D\varphi d\rho d\zeta + 2 \int_0^l \int_{\rho_0}^{1} \frac{1}{\rho} f D\varphi d\rho d\zeta \tag{2.10}$$

be a minimum.

Let us determine the nature of the convergence of the approximate solution to the exact one. (In the sequel, the integration with respect to ζ will be between the limit 0 and l, and with respect to ρ from ρ_0 to 1; we shall not indicate these limits from now on.) If φ_0 is a solution of the variational problem, while φ_n is a minimizing sequence, then by (1.8) it follows from the condition

$$J(\varphi_n) = \|\varphi_n - \varphi_0\|_A{}^2 - \|\varphi_0\|_A{}^2 \to -\|\varphi_0\|_A{}^2 \tag{2.11}$$

that $\|\varphi_n - \varphi_0\|_A{}^2 \to 0$, where $\|\varphi\|_A{}^2 = (A\varphi, B\varphi)$, and is equal to (2.7). Substituting $\varphi_n - \varphi_0$ into (2.7) in place of φ, and taking into account (2.9), we obtain

$$\iint \frac{1}{\rho}\left(\frac{\partial D\varphi_n}{\partial\rho} - \frac{\partial D\varphi_0}{\partial\rho}\right)^2 d\rho\, d\zeta \to 0, \quad \iint \frac{1}{\rho}\left(\frac{\partial D\varphi_n}{\partial\zeta} - \frac{\partial D\varphi_0}{\partial\zeta}\right)^2 d\rho\, d\zeta \to 0$$
$$\iint \frac{1}{\rho}\left(\frac{\partial^3\varphi_n}{\partial\rho\partial\zeta^2} - \frac{\partial^3\varphi_0}{\partial\rho\,\partial\zeta^2}\right)^2 d\rho\, d\zeta \to 0 \tag{2.12}$$

i.e. the functions $\rho^{-1}\partial D\varphi_n/\partial\rho$, $\rho^{-1}\partial D\varphi_n/\partial\zeta$, $\rho^{-1}\partial^3\varphi_n/\partial\rho\partial\zeta^2$ converge to the corresponding functions of the exact solution in the mean and with the weight ρ.

Let us prove the convergence in the mean also for the functions $\rho^{-2}\partial^2\varphi_n/\partial\zeta^2$, $\rho^{-2}D\varphi_n$. We consider the identity

$$\frac{1}{2}\iint \frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial^2\varphi}{\partial\zeta^2}\right)^2 d\rho d\zeta = \frac{1}{2}\int\left[\left(\frac{\partial^2\varphi(1)}{\partial\zeta^2}\right)^2 - \frac{1}{\rho_0{}^2}\left(\frac{\partial^2\varphi(\rho_0)}{\partial\zeta^2}\right)^2\right]d\zeta =$$
$$= \iint \frac{1}{\rho^2}\frac{\partial^3\varphi}{\partial\rho\,\partial\zeta^2}\frac{\partial^2\varphi}{\partial\zeta^2}\, d\rho\, d\zeta - \iint\frac{1}{\rho^3}\left(\frac{\partial^2\varphi}{\partial\zeta^2}\right)^2 d\rho\, d\zeta$$

Making use of the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we obtain

$$\left(\iint\frac{1}{\rho^3}\left(\frac{\partial^2\varphi}{\partial\zeta^2}\right)^2 d\rho d\zeta\right)^2 \leqslant 2\left(\iint\frac{1}{\rho^2}\frac{\partial^3\varphi}{\partial\rho\,\partial\zeta^2}\frac{\partial^2\varphi}{\partial\zeta^2}\,d\rho d\zeta\right)^2 + \frac{1}{2}\left(\int\left[\left(\frac{\partial^2\varphi(1)}{\partial\zeta^2}\right)^2 - \frac{1}{\rho_0{}^2}\left(\frac{\partial^2\varphi(\rho_0)}{\partial\zeta^2}\right)^2\right]d\zeta\right)^2$$

If the first integral on the right-hand side is larger than the second one, then, by using the Cauchy-Buniakovskii inequality, we obtain

$$\iint\left(\frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\zeta^2}\right)^2\rho d\rho\, d\zeta \leqslant \sqrt{\frac{5}{2}}\iint\frac{1}{\rho}\left(\frac{\partial^3\varphi}{\partial\rho\,\partial\zeta^2}\right)^2 d\rho\, d\zeta \tag{2.13}$$

If the second integral is larger than the first one, then the validity of the inequality (2.13) is obvious in view of (2.9). Let us consider still another identity. In view of (2.3) we have

$$\frac{1}{2}\iint\frac{\partial}{\partial\rho}\left(\frac{D\varphi}{\rho}\right)^2 d\rho\, d\zeta = \iint\frac{D\varphi}{\rho^2}\frac{\partial D\varphi}{\partial\rho}\, d\rho\, d\zeta - \iint\left(\frac{D\varphi}{\rho^2}\right)^2\rho d\rho d\zeta = 0$$

Applying Hölder's inequality we obtain

$$\left(\iint\left(\frac{D\varphi}{\rho^2}\right)^2\rho d\rho\, d\zeta\right)^2 \leqslant \iint\left(\frac{D\varphi}{\rho^2}\right)^2\rho d\rho d\zeta\iint\left(\frac{1}{\rho}\frac{\partial D\varphi}{\partial\rho}\right)^2\rho d\rho\, d\zeta$$

Whence

$$\iint \left(\frac{D\varphi}{\rho^2}\right)^2 \rho d\rho d\zeta \leqslant \iint \frac{1}{\rho}\left(\frac{\partial D\varphi}{\partial \rho}\right)^2 d\rho\, d\zeta \tag{2.14}$$

Substituting $\varphi_0 - \varphi_n$ in (2.13) and (2.14) in place of φ, and taking into account (2.12), we find that the functions $\rho^{-2}\partial^2\varphi_n/\partial\zeta^2$, $\rho^{-2} D\varphi_n$ converge in the mean to the functions $\rho^{-2}\partial^2\varphi_0/\partial\zeta^2$, and $\rho^{-2} D\varphi_0$, respectively.

All of these quantities, of which the convergence in the mean has been proved, and only these quantities, occur in the expressions for the stress in (2.1). Therefore, it is now easy to establish the convergence in the mean of the stresses themselves. Let σ_{nr} and σ_{0r}, defined by the formulas (2.1), correspond to the approximate φ_n, and to the exact φ_0 solutions. Then

$$\iint (\sigma_{0r}-\sigma_{nr})^2 \rho d\rho d\zeta = \iint\left[\frac{1}{\rho}\frac{\partial^3(\varphi_0-\varphi_n)}{\partial\rho\,\partial\zeta^2} - \frac{1-\mu}{\rho^2}\frac{\partial^2(\varphi_0-\varphi_n)}{\partial\zeta^2} + \frac{\mu}{\rho^2}D(\varphi_n-\varphi_0)\right]^2 \rho d\rho d\zeta$$

Making use of Minkovskii's inequality [9], we obtain

$$\|\sigma_{0r}-\sigma_{nr}\| = \left(\iint(\sigma_{0r}-\sigma_{nr})^2\rho d\rho\, d\zeta\right)^{1/2} \leqslant \left(\iint\frac{1}{\rho}\left(\frac{\partial^3(\varphi_0-\varphi_n)}{\partial\rho\,\partial\zeta^2}\right)^2 d\rho d\zeta\right)^{1/2} +$$
$$+\left(\iint\frac{(1-\mu)^2}{\rho^3}\left(\frac{\partial^2(\varphi_0-\varphi_n)}{\partial\zeta^2}\right)^2 d\rho d\zeta\right)^{1/2} + \left(\iint\frac{\mu^2}{\rho^3}(D(\varphi_0-\varphi_n))^2 d\rho d\zeta\right)^{1/2}$$

The convergence of the quantities on the right-hand side has been proved, and the convergence in the mean with weight ρ of the functions σ_{nr} and σ_{0r} has thereby also been established. The convergence for the remaining stresses is proved in an analogous manner.

Since every term on the right-hand side of the inequality is less than $2\|\varphi_n - \varphi_0\|_A$ because of the relations (2.12) to (2.14), it follows that the deviation in the mean of the approximate values of the stresses from the exact values can be appraised as $\|\varphi_n - \varphi_0\|_A$.

From (2.11) it is obvious that $\|\varphi_n - \varphi_0\|_A^2 = J(\varphi_n) + \|\varphi_0\|_A^2$, and if there exists a quantity δ such that $\delta \leqslant -\|\varphi_0\|_A^2$, or $-\delta \geqslant \|\varphi_0\|_A^2$, then

$$\|\varphi_n - \varphi_0\|_A \leqslant \sqrt{J(\varphi_n)-\delta} \tag{2.15}$$

This estimate will become more precise if δ comes closer to $-\|\varphi_0\|_A^2$. It is natural to try to determine a functional which would converge to min $J(\varphi) = -\|\varphi_0\|_A^2$ from below, i.e. one whose maximum would be the min $J(\varphi)$. The construction of this functional will be carried out in accordance with a general idea of Friedrichs, extended by Sobolev [11] to partial derivatives of high order, whose solution is connected to the problem on the minimal functional. Let us apply Sobolev's method to the functional (2.10). Let us represent (2.10) in the form

$$J(\varphi) = \Phi(\varphi) + 2\iint \frac{1}{\rho} f D\varphi \, d\rho \, d\zeta \tag{2.16}$$

where $\Phi(\varphi)$ is given by (2.7).

The equation (2.1) will be a natural condition for a minimum of the functional $J(\varphi)$ [10]. Let us make the change of variables $D\varphi = u$, $\partial^2\varphi/\partial\zeta^2 = v$ in $\Phi(\varphi)$. Then the formula (2.7) yields

$$\Phi(u, v) = \iint \frac{1}{\rho}\left[\left(\frac{\partial u}{\partial \rho}\right)^2 + 2\left(\frac{\partial u}{\partial \zeta}\right)^2 + \left(\frac{\partial v}{\partial \rho}\right)^2\right] d\rho d\zeta - (1-\mu)\int\left[v^2(1) - \frac{1}{\rho_0{}^2} v^2(\rho_0)\right] d\zeta \tag{2.17}$$

The equation (2.1) will take on the form

$$Du + 2\frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 v}{\partial \zeta^2} = f \tag{2.18}$$

If φ_0 is now the exact solution of the problem on the minimum of the functional (2.16), then, setting $u_0 = D\varphi_0$, $v_0 = \partial^2\varphi_0/\partial\zeta^2$, we obtain the following expressions for u_0 and v_0 on the basis of (2.3)

$$u_0 = 0, \quad \frac{\partial v_0}{\partial \rho} = \frac{1-\mu}{\rho} v_0 \quad \text{if} \quad \rho = \begin{cases}\rho_0 \\ 1\end{cases} \quad u_0 = 0, \quad \frac{\partial u_0}{\partial \zeta} = 0 \quad \text{if} \quad \zeta = \begin{cases} 0 \\ l\end{cases} \tag{2.19}$$

In view of the inequality (2.9), we have $\Phi(u, v) \geqslant 0$. Replacing u and v in (2.17) by the differences $u_0 - u_n$ and $v_0 - v_n$, where u_n and v_n are arbitrary functions, we find that $\Phi(u_0 - u_n, v_0 - v_n) \geqslant 0$. If, for example, the following boundary conditions

$$u_n = 0, \qquad \frac{\partial v_n}{\partial \rho} = \frac{1-\mu}{\rho} v_n \quad \text{if} \quad \rho = \rho_0, \ \rho = 1 \tag{2.20}$$

are satisfied, then $\Phi(u_0 - u_n, v_0 - v_n) = 0$, if, and only if, $u_n = u_0$, $v_n = v_0$.

Let us subject u_n and v_n to the additional conditions (2.18), which is natural to do when $u_n \to u_0$ and $v_n \to v_0$. Then

$$\Phi(u_0 - u_n, v_0 - v_n) = \Phi(u_0, v_0) + \Phi(u_n, v_n) - F(u_0, v_0, u_n, v_n)$$

On the basis of (2.19), and the identity $\partial^2 u_0/\partial\zeta^2 = Dv_0$, the F in this last displayed equation equation can be found by integration by parts in the form

$$F = -2\iint \frac{u_0}{\rho}\left(Du_n + 2\frac{\partial^2 u_n}{\partial \zeta^2} + \frac{\partial^2 v_n}{\partial \zeta^2}\right) d\rho \, d\zeta = -2\iint \frac{1}{\rho} D\varphi_0 f \, d\rho \, d\zeta$$

Now $\Phi(u_0, v_0) - F = J(\varphi_0) = \min J(\varphi)$, and since $\Phi(u_n, v_n) + \min J(\varphi) \geqslant 0$, it follows that

$$\max[-\Phi(u_n, v_n)] = \min J(\varphi) \quad \text{for} \quad u_n \to u_0, \ v_n \to v_0$$

Hence it is obvious that

$$\delta = -\Phi(u_n, v_n)$$

If one sets $u_n = D\psi$, $v_n = \partial^2\psi/\partial\zeta^2$, one obtains $\delta = -\Phi(\psi)$, where ψ is a solution of the equation (2.1), which in accordance with (2.20) satisfies the conditions

$$\rho = \rho_0, \qquad \rho = 1, \qquad D\psi = 0, \qquad \frac{\partial^3\psi}{\partial\rho\,\partial\zeta^2} = \frac{1-\mu}{\rho}\frac{\partial^2\psi}{\partial\zeta^2} \tag{2.21}$$

One can show that all stresses satisfy at least the fairly rough inequality

$$\|\sigma_n - \sigma_0\| \leqslant (1+\mu)\sqrt{J(\varphi_n) - \delta},$$

if none of them is identically zero.

3. Let us apply Galerkin's method (1.13) to the solution of the axisymmetric problem. We select a system of functions in the form $\varphi_{nm} = x_n(\rho)\,y_m(\zeta)$. For the purpose of obtaining a simpler system of equations (1.13), we determine the $x_n(\rho)$ as the characteristic functions of the equation

$$D(\rho Z_n) = -\lambda_n^2 \rho Z_n, \quad Z_n = Dx_n, \quad \rho = \rho_0, \quad \rho = 1, \quad Z_n = 0$$

Then the characteristic functions, which have been orthonormalized with the weight ρ, will be of the form

$$Z_n = Dx_n = c_n Z_1(\lambda_n\rho), \qquad Z_1(\lambda_n\rho) = N_1(\lambda_n)J_1(\lambda_n\rho) - J_1(\lambda_n)N_1(\lambda_n\rho)$$

$$c_n = \sqrt{2}\,[Z_0^2(\lambda_n) - \rho_0^2 Z_0^2(\lambda_n\rho_0)]^{-1/2}, \qquad Z_0(\lambda_n\rho) = N_1(\lambda_n)J_0(\lambda_n\rho) - J_1(\lambda_n)N_0(\lambda_n\rho)$$

Here J_0, J_1, N_0 and N_1 are Bessel functions of the first and second kind of the zero-th and first order; λ_n is a root of the equation

$$Z_1(\lambda_n\rho_0) = 0 \tag{3.1}$$

As is known [12], the indicated system of functions will be complete in $L_2(\rho)$. By integration we obtain

$$x_n(\rho) = -\frac{c_n}{\lambda_n^2}\rho Z_1(\lambda_n\rho) + \frac{1}{1-\rho_0^2}\left(\rho_0^2\frac{\delta_n}{1-\mu} + \rho^2\frac{\gamma_n}{1+\mu}\right)$$

$$\delta_n = \frac{c_n}{\lambda_n}[Z_0(\lambda_n) - Z_0(\lambda_n\rho_0)], \qquad \gamma_n = \frac{c_n}{\lambda_n}[Z_0(\lambda_n) - \rho_0^2 Z_0(\lambda_n\rho_0)]$$

The function x_n satisfies the conditions

$$\rho = \rho_0, \quad \rho = 1, \quad Dx_n = 0, \quad \rho\, \partial x_n / \partial \rho = (1 - \mu)\, x_n$$

We determine the y_m as orthonormalized characteristic functions of the equation

$$y_m^{\mathrm{IV}} = \nu_m^4 y_m, \quad \zeta = 0, \quad \zeta = l, \quad y_m = 0 \quad y_m' = 0$$

Thus we find

$$y_m = \frac{1}{\sqrt{l}} \left[\cosh \nu_m \zeta - \cos \nu_m \zeta - \frac{\cosh \nu_m l - \cos \nu_m l}{\sinh \nu_m l - \sin \nu_m l} (\sinh \nu_m \zeta - \sin \nu_m \zeta) \right]$$

where ν_m is a root of the equation $\cosh \nu_m l \cos \nu_m l = 1$; hence, if $\nu_1 l = 4.730$, $\nu_2 l = 7.853$, the remaining roots can be determined with sufficient accuracy by the formula

$$\nu_m l = m\pi + \pi / 2, \; m = 3, 4 \ldots$$

As is known [10], the system of functions y_m is complete in L_2.

By the definition of x_n and y_m, the system of functions φ_{nm} will be a complete system of elements in M in the subspace H_B. From (2.5) and (2.7) it follows that $H_A \subset H_B$, since the elements of H_A have to have derivatives of higher order than those of the elements of H_B. Hence, the system of functions is also complete in H_A. Furthermore, this system is orthonormalized in H_B, and it is, therefore, strongly minimal in H_A, and the method of Ritz (1.12), and the equivalent method of Galerkin (1.13), applied to this problem below, will be stable [13].

We shall look for a solution in the form

$$\varphi_N = \sum_{n,m=1}^{N} A_{nm}\, x_n(\rho)\, y_m(\zeta)$$

The system of equations (1.13) will have the following form in this case (3.2)

$$\left(\lambda_n^2 + \frac{\nu_m^4}{\lambda_n^2} \right) A_{nm} + \sum_{k=1}^{N} \alpha_{km} A_{nk} + \nu_m^4 \sum_{i=1}^{N} \beta_{ni} A_{im} = f_{n,m} \qquad (n, m = 1, \ldots, N)$$

Here

$$f_{n,m} = -c_n \int_0^l \int_{\rho_0}^1 Z_1(\lambda_n \rho)\, y_m(\zeta)\, f(\rho, \zeta)\, d\rho\, d\zeta$$

$$\beta_{ni} = \frac{1}{1-\rho_0^2}\left(\rho_0^2 \frac{\delta_n \delta_i}{1-\mu} + \frac{\gamma_n \gamma_i}{1+\mu}\right) = \beta_{in}$$

$$\alpha_{mk} = \alpha_{km} = \pm \frac{16\, \nu_k^2 \nu_m^2}{l(\nu_k^4 - \nu_m^4)}\left(\nu_k \frac{\sin \nu_k l}{1 \pm \cos \nu_k l} - \nu_m \frac{\sin \nu_m l}{1 \pm \cos \nu_m l}\right)$$

$$\alpha_{mm} = \pm \frac{2\nu_m \sin \nu_m l}{l(1 \pm \cos \nu_m l)}\left(2 \pm \nu_m l \frac{\sin \nu_m l}{1 \pm \cos \nu_m l}\right)$$

The plus sign must be used in the case when k and m are odd, while the minus sign applies when k and m are even. If, however, one of the number k or m is odd while the other one is even, then $\alpha_{km} = 0$. This latter condition leads to the result that the system of equations (3.2) separates into two independent systems of equations with a smaller number of unknown quantities. This simplifies the practical application of the system of equations (3.2) considerably. It is easy to see that the matrix of the coefficients of the A_{ik} is symmetric.

Let us consider an example on the application of the system (3.2). Suppose that on the inner side wall of the cylinder there is given a constant temperature t_0, and that on the outer surface of the side wall the temperature is zero.

In this case a solution of the equation of heat conduction will be $t = t_0 \ln \rho / \ln \rho_0$. By the formulas (2.4) and (3.2) we obtain $f = -E\alpha t_0 / (1 - \mu) \ln \rho_0$

$$f_{nm} = \delta_n \frac{4 \sin \nu_m l}{\nu_m \sqrt{l}\,(1 + \cos \nu_m l)} \frac{E\alpha t_0}{(1-\mu)\ln \rho_0}, \quad m = 2k+1, \quad f_{nm} = 0, \quad m = 2k$$
$$(k = 0, 1, 2 \ldots)$$

Suppose that the ratio of the inner radius to the outer radius is $\rho_0 = 0.5$, and that the length of the cylinder $L = 2R$, i.e. $l = 2$. We shall perform the analysis under the assumption that $N = 2$. We determine the roots of the equation (3.1) from tables [14]. We find that $\lambda_1 = 6.393$, $\lambda_2 = 12.625$. Solving the system of equations (3.2), we obtain

$$A_{11} = -0.0259\, E\alpha t_0, \qquad A_{21} = 0.00107\, E\alpha t_0, \qquad A_{12} = 0, \qquad A_{22} = 0$$

and, therefore

$$\varphi_N = [A_{11} x_1(\rho) + A_{21} x_2(\rho)]\, y_1(\zeta)$$

Let us compute the max σ_z and max σ_θ, when $\rho = \rho_0$, where they take on small values

$$\max \sigma_z = \sigma_z(0.5,1) = -0.629\, E\alpha t_0, \qquad \max \sigma_\theta = \sigma_\theta(0.5,1) = -0.426\, E\alpha t_0$$

The values of the stresses when $\rho = \rho_0 = 0.5$, computed by means of formulas for an infinitely long cylinder [8], will be equal to $\max \sigma_z = \max \sigma_\theta = -0.874\, E\alpha t_0$, which shows that the max σ_θ is almost twice as large as the value given above. The function

$$\psi = \left[\left(\frac{\rho_0^2 \ln \rho_0}{16(1-\rho_0^2)} - \frac{3}{64}\right)\rho^4 - \frac{\rho_0^2 \ln \rho_0}{4(1-\rho_0^2)}\rho^2 \ln \rho + \frac{1}{16}\rho^4 \ln \rho\right] f$$

is a solution of the equation (2.1) for the given example, which satisfies the conditions (2.21). Hence, in accordance with (2.10) and (2.17) we obtain

$$\sqrt{J(\varphi_n) + \Phi(\psi)} = \sqrt{J(\varphi_n) - \delta} = 0.065\, E\alpha t_0$$

i.e. the mean error for the stresses does not exceed 20% of their maximum value.

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